Optomechanics III

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Problem I: The input–output relation

The point of this exercise is to familiarise the student with how unitary dynamics can become effectively non-unitary when we concentrate on a "small" system interacting with a "large" environment. I have decided to concentrate on a fundamental quantum-optics exercise rather than a purely optomechanics one, because I think that in the context of this Winter School, such an exercise would be very useful.

In Lecture II, I introduced a term $-\sqrt{2\kappa}\hat{a}_{\text{in}}$ into the equation of motion for $\hat{a}$, calling it simply "noise." I did not explain where this term comes from or how to obtain it. Indeed, it is not trivial to see how it emerges from the master equation description I used in Lecture I, but we can obtain it otherwise, in the process learning a bit about the nature of open quantum systems; this is a fundamental issue in optomechanics.

The interaction of the cavity field with its electromagnetic environment can be modelled using the Hamiltonian

$$\hat{H} = \hbar\omega_{\text{c}}\hat{a}^\dagger\hat{a} + \hbar\int \mathrm{d}\omega\ \omega\ \hat{a}_\omega^\dagger\hat{a}_\omega + i\hbar\sqrt{\frac{\kappa}{\pi}}\int \mathrm{d}\omega\ \left(\hat{a}\ \hat{a}_\omega^\dagger - \hat{a}^\dagger\ \hat{a}_\omega\right)$$

Here, the electromagnetic environment is modelled as an infinite collection of harmonic oscillators, spanning the entire frequency range. Each of these oscillators is coupled with coupling strength $\sqrt{\kappa/\pi}$ to the cavity field operator.

Task 1

The first task is fairly straightforward: Derive the Heisenberg equations of motion for $\hat{a}$ and $\hat{a}_\omega$.

Solution…

As usual, we use the commutator of the operator with the Hamiltonian to obtain

$$\dot{\hat{a}} = -i\omega_{\text{c}}\hat{a} - \sqrt{\frac{\kappa}{\pi}}\int \hat{a}_\omega\ \mathrm{d}\omega$$

$$\dot{\hat{a}}_\omega = -i\omega\hat{a}_\omega + \sqrt{\frac{\kappa}{\pi}}\hat{a}$$

Task 2

Next, formally integrate the equation of motion for $\hat{a}_\omega$, from some time t_0 in the distant past to the current time t, and substitute into that for $\hat{a}$. Evaluate the integrals that appear in $\dot{\hat{a}}$.

To do this, define the input field $\hat{a}_{\text{in}}(t) = \frac{1}{\sqrt{2\pi}}\int \mathrm{d}\omega\ e^{-i\omega(t-t_0)}\hat{a}_\omega(t_0)$. Then, use $\int \mathrm{d}\omega\ e^{-i\omega t} = 2\pi\ \delta(t)$ and $\int_{t_0}^{t} \mathrm{d}\tau\ \hat{a}(\tau)\delta(t-t_0) = \frac{1}{2}\hat{a}(t)$, where the factor of $\frac{1}{2}$ comes in because the delta function is peaked at the end of the integration interval.

Solution…

From the second equation above, it is easy to obtain

$$\hat{a}_\omega(t) = e^{-i\omega(t-t_0)}\hat{a}_\omega(t_0) + \sqrt{\frac{\kappa}{\pi}}\int_{t_0}^{t} e^{-i\omega(t-\tau)}\hat{a}(\tau)\,\mathrm{d}\tau$$

Next, as instructed, we substitute this into the other equation of motion:

$$\dot{\hat{a}} = -i\omega_\mathrm{c}\hat{a} - \sqrt{\frac{\kappa}{\pi}}\int\left[e^{-i\omega(t-t_0)}\hat{a}_\omega(t_0) + \sqrt{\frac{\kappa}{\pi}}\int_{t_0}^{t} e^{-i\omega(t-\tau)}\hat{a}(\tau)\,\mathrm{d}\tau\right]\mathrm{d}\omega$$

How should we proceed? For the first term in brackets, we simply use the definition of $\hat{a}_\mathrm{in}$ as given in the hint. For the second term, we switch around the order of integration and evaluate the ω integral first, not forgetting the factor of $\frac{1}{2}$:

$$\int\int_{t_0}^{t} e^{-i\omega(t-\tau)}\hat{a}(\tau)\,\mathrm{d}\tau\;\mathrm{d}\omega = \int_{t_0}^{t}\hat{a}(\tau)\int e^{-i\omega(t-\tau)}\,\mathrm{d}\omega\;\mathrm{d}\tau = \pi\,\hat{a}(t)$$

With these simplifications, we obtain

$$\dot{\hat{a}} = -i\omega_\mathrm{c}\hat{a} - \sqrt{\frac{\kappa}{\pi}}\left(\sqrt{2\pi}\,\hat{a}_\mathrm{in} + \sqrt{\pi}\,\kappa\,\hat{a}\right)$$

Thus,

$$\dot{\hat{a}} = -(i\omega_\mathrm{c} + \kappa)\hat{a} - \sqrt{2\kappa}\,\hat{a}_\mathrm{in}$$

Task 3

We have defined $\hat{a}_\mathrm{in}$ using a time in the distant past. Define a time in the *distant future*, t_1, and the *output field*: $\hat{a}_\mathrm{out}(t) = \frac{1}{\sqrt{2\pi}}\int \mathrm{d}\omega\; e^{-i\omega(t-t_1)}\hat{a}_\omega(t_1)$. What I'd like you to do now is to redo the above exercise, but integrate the differential equation backward in time from t_1 to t, in order to derive an equation of motion for $\hat{a}$ in terms of $\hat{a}_\mathrm{out}$.

Solution…

First,

$$\hat{a}_\omega(t) = e^{-i\omega(t-t_1)}\hat{a}_\omega(t_1) + \sqrt{\frac{\kappa}{\pi}}\int_{t_1}^{t} e^{-i\omega(t-\tau)}\hat{a}(\tau)\,\mathrm{d}\tau$$

This yields

$$\dot{\hat{a}} = -i\omega_\mathrm{c}\hat{a} - \sqrt{\frac{\kappa}{\pi}}\int\left[e^{-i\omega(t-t_1)}\hat{a}_\omega(t_1) + \sqrt{\frac{\kappa}{\pi}}\int_{t_1}^{t} e^{-i\omega(t-\tau)}\hat{a}(\tau)\,\mathrm{d}\tau\right]\mathrm{d}\omega$$

Proceeding as before, we obtain

$$\dot{\hat{a}} = -i\omega_\mathrm{c}\hat{a} - \sqrt{\frac{\kappa}{\pi}}\left(\sqrt{2\pi}\,\hat{a}_\mathrm{out} - \sqrt{\pi}\,\kappa\,\hat{a}\right)$$

where the minus sign in the last term appears because $t_1 > t$. Simplifying by collecting terms,

$$\dot{\hat{a}} = -(i\omega_\mathrm{c} - \kappa)\hat{a} - \sqrt{2\kappa}\,\hat{a}_\mathrm{out}$$

Task 4

Back to a simple question. Equating the two expressions for $\dot{\hat{a}}$, obtain the celebrated input–output relation, $\hat{a}_\mathrm{out} = \hat{a}_\mathrm{in} + \sqrt{2\kappa}\,\hat{a}$.

Solution…

There is nothing too tricky here:

$$-(i\omega_{\mathrm{c}}-\kappa)\hat{a}-\sqrt{2\kappa}\,\hat{a}_{\mathrm{out}}=-(i\omega_{\mathrm{c}}+\kappa)\hat{a}-\sqrt{2\kappa}\,\hat{a}_{\mathrm{in}}$$

Therefore,

$$\hat{a}_{\mathrm{out}}=\hat{a}_{\mathrm{in}}+\sqrt{2\kappa}\,\hat{a}$$

Task 5

We have not mentioned anything about the properties of $\hat{a}_\omega$. Indeed, being separate Bosonic fields, they satisfy the commutation relation $[\hat{a}_\omega,\hat{a}^\dagger_{\omega'}]=\delta(\omega-\omega')$. Show that this implies that $[\hat{a}_{\mathrm{in}}(t),\hat{a}^\dagger_{\mathrm{in}}(t')]=\delta(t-t')$.

Solution…

This is a question that can be solved by a straightforward evaluation. We have

$$[\hat{a}_{\mathrm{in}}(t),\hat{a}^\dagger_{\mathrm{in}}(t')]=\frac{1}{2\pi}\int\int e^{-i\omega(t-t_0)}e^{i\omega'(t'-t_0)}\,[\hat{a}_\omega,\hat{a}^\dagger_{\omega'}]\,\mathrm{d}\omega'\,\mathrm{d}\omega$$

Using the given commutation relation, we can reduce the double-integral to a single integral:

$$[\hat{a}_{\mathrm{in}}(t),\hat{a}^\dagger_{\mathrm{in}}(t')]=\frac{1}{2\pi}\int e^{-i\omega(t-t')}\,\mathrm{d}\omega$$

This is again an integral that we have seen before, and yields

$$[\hat{a}_{\mathrm{in}}(t),\hat{a}^\dagger_{\mathrm{in}}(t')]=\delta(t-t')$$

Task 6

In Lecture II, I claimed that the noise term was necessary to ensure that the Bosonic commutation relation for $\hat{a}$ holds. Assuming that the equal-time commutation relation $[\hat{a}_{\mathrm{out}}(t),\hat{a}^\dagger(t)]=-[\hat{a}_{\mathrm{in}}(t),\hat{a}^\dagger(t)]$ holds, show that $[\hat{a}(t),\hat{a}^\dagger(t)]=1$ for all time.

Solution…

Using the input–output relation,

$$[\hat{a}_{\mathrm{out}}(t),\hat{a}^\dagger(t)]=[\hat{a}_{\mathrm{in}}(t),\hat{a}^\dagger(t)]+\sqrt{2\kappa}[\hat{a}(t),\hat{a}^\dagger(t)]$$

Thus,

$$[\hat{a}_{\mathrm{in}}(t),\hat{a}^\dagger(t)]=\tfrac{1}{2}\sqrt{2\kappa}[\hat{a}(t),\hat{a}^\dagger(t)]$$

What is the time derivative of $[\hat{a}(t),\hat{a}^\dagger(t)]$? It is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}t}[\hat{a}(t),\hat{a}^\dagger(t)]=[\dot{\hat{a}}(t),\hat{a}^\dagger(t)]+[\hat{a}(t),\dot{\hat{a}}^\dagger(t)]$$

Substituting the correct equation of motion,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[\hat{a}(t),\hat{a}^\dagger(t)]&=[-(i\omega_{\mathrm{c}}+\kappa)\hat{a}(t)-\sqrt{2\kappa}\,\hat{a}_{\mathrm{in}}(t),\hat{a}^\dagger(t)]\\&\quad+[\hat{a}(t),-(-i\omega_{\mathrm{c}}+\kappa)\hat{a}^\dagger(t)-\sqrt{2\kappa}\,\hat{a}^\dagger_{\mathrm{in}}(t)]\end{aligned}$$

Simplifying,

$$\frac{\mathrm{d}}{\mathrm{d}t}[\hat{a}(t), \hat{a}^\dagger(t)] = -2\kappa[\hat{a}(t), \hat{a}^\dagger(t)] - \sqrt{2\kappa}\left([\hat{a}_{\mathrm{in}}(t), \hat{a}^\dagger(t)] + [\hat{a}(t), \hat{a}_{\mathrm{in}}^\dagger(t)]\right) = 0$$

Taking $t \to -\infty$, i.e., before any interaction of the cavity field with the outside world, we know that

$$[\hat{a}(t), \hat{a}^\dagger(t)] = 1 \; (t \to -\infty)$$

Therefore, this commutation relation must hold for all time.

Notes

The relation obtained in Task 4 is extremely useful in practice because it tells us how we can extract information from the cavity simply by monitoring the output field. This may sound obvious, but the input–output relation gives it a quantitative meaning and makes the relationship between "what enters," "what lies inside," and "what leaks out of" the cavity concrete.

Let us also reflect on the equation of motion for $\hat{a}$ in one of its versions:

$$\dot{\hat{a}} = -(i\omega_{\mathrm{c}} + \kappa)\hat{a} - \sqrt{2\kappa}\,\hat{a}_{\mathrm{in}}$$

Note how the appearance of κ has changed the dynamics of $\hat{a}$ from unitary to non-unitary. In other words, if the cavity field is in a single-photon state, for example, that state may spontaneously and irreversibly decay to the vacuum state. A (very) good student might object to this statement, claiming that the photon does not disappear, but is to be found in one of the infinitely many $\hat{a}_\omega$ modes. This is indeed the correct interpretation; since the overall dynamics is governed by a Hamiltonian and nothing else, it must be unitary, and the photon cannot simply disappear. Should one take a *finite* number of bath modes, one would indeed find recurrences, where the photon disappears from the cavity mode only to come back at a later time. In the limit of infinitely many densely packed bath modes, this recurrence time diverges and the photon never comes back. As far as the cavity field is concerned, therefore, the photon disappears irreversibly and the dynamics is non-unitary.

Further rigour and detail are to be found in Chapters 3 and 5 of [C. W. Gardiner and P. Zoller, *Quantum Noise* (Springer, Berlin, Heidelberg, 2004)]

Problem II: Limiting occupation number

In the second lecture I mentioned that the "noise terms" limit the cooling of the mechanical element to nonzero temperatures. Throughout this problem, we will obtain an expression for the steady-state occupation number of the mechanical system, directly related to its effective temperature, under the cooling mechanism discussed in the lecture.

Task 1

We start from the two linear equations governing the evolution of the optical and mechanical systems:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t) = (i\Delta - \kappa)\hat{a}(t) + iG\left[\hat{b}(t) + \hat{b}^\dagger(t)\right] - \sqrt{2\kappa}\hat{a}_{\mathrm{in}}(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{b}(t) = -(i\omega_{\mathrm{m}} + \gamma)\hat{b}(t) + iG\left[\hat{a}(t) + \hat{a}^\dagger(t)\right] - \sqrt{2\gamma}\hat{b}_{\mathrm{in}}(t)$$

In this and the following tasks, we shall perform a more careful adiabatic elimination of $\hat{a}$, keeping track of the noise terms. Since the non-noise terms have already been covered in the derivation outlined during the lecture, let us focus on the noise now. First, formally solve the equation for $\hat{a}$, integrating over time from $t = -\infty$ and keeping track of the contribution due to $\hat{a}_{\mathrm{in}}$. Next, substitute this relation into the equation for $\hat{b}$. For the non-noise terms, recall from the lecture that

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{b}(t) = -\left[i\left(\omega_{\mathrm{m}} + \omega_{\mathrm{opt}}\right) + \left(\gamma + \gamma_{\mathrm{opt}}\right)\right]\hat{b}(t) + \cdots$$

where $\omega_{\mathrm{opt}} = G^2/2\omega_{\mathrm{m}}$ and $\gamma_{\mathrm{opt}} = G^2/\kappa$. We will be working in the resolved-sideband limit, i.e., $\omega_{\mathrm{m}} \gg \kappa$, so that we will approximate $\omega_{\mathrm{opt}} \approx 0$ and concentrate on the cooling. Your first task is to fill in the "…" in the above equation; show that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{b}(t) = \cdots - iG\sqrt{2\kappa}\left[\int_{-\infty}^{t} \mathrm{d}\tau\, e^{(i\Delta-\kappa)(t-\tau)}\hat{a}_{\mathrm{in}}(\tau) + \int_{-\infty}^{t} \mathrm{d}\tau\, e^{(-i\Delta-\kappa)(t-\tau)}\hat{a}_{\mathrm{in}}^\dagger(\tau)\right] - \sqrt{2\gamma}\hat{b}_{\mathrm{in}}(t)$$

Solution…

The first part is fairly straightforward, and is a matter of simply remembering how to solve a first-order differential equation (at least formally):

$$\hat{a}(t) = iG\int_{-\infty}^{t} \mathrm{d}\tau\, e^{(i\Delta-\kappa)(t-\tau)}\left[\hat{b}(t) + \hat{b}^\dagger(t)\right] - \sqrt{2\kappa}\int_{-\infty}^{t} \mathrm{d}\tau\, e^{(i\Delta-\kappa)(t-\tau)}\hat{a}_{\mathrm{in}}(\tau)$$

We can now substitute this expression for $\hat{a}$ into the equation for $\hat{b}$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{b}(t) = &-\left[i\omega_{\mathrm{m}} + \left(\gamma + \gamma_{\mathrm{opt}}\right)\right]\hat{b}(t) \\ &- iG\sqrt{2\kappa}\left[\int_{-\infty}^{t} \mathrm{d}\tau\, e^{(i\Delta-\kappa)(t-\tau)}\hat{a}_{\mathrm{in}}(\tau) + \int_{-\infty}^{t} \mathrm{d}\tau\, e^{(-i\Delta-\kappa)(t-\tau)}\hat{a}_{\mathrm{in}}^\dagger(\tau)\right] - \sqrt{2\gamma}\hat{b}_{\mathrm{in}}(t)\end{aligned}$$

Task 2

In a similar spirit as when you solved the equation for $\hat{a}$ in completing Task 1, you may formally solve the above equation for $\hat{b}$. This will give rise to two sets of terms, one related to $\hat{a}_{\mathrm{in}}$ and one to $\hat{b}_{\mathrm{in}}$. The latter will be easier to handle, so let's start with that. Considering only the $\hat{b}_{\mathrm{in}}$ term in the above differential equation, show that $\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle = \gamma/(\gamma+\gamma_{\text{opt}})\, n$, where n is the mean number of phonons in the bath that the mechanical oscillator is coupled to. You will need to use the fact that

$$\langle\hat{b}_{\text{in}}^\dagger(t)\hat{b}_{\text{in}}(t')\rangle = n\,\delta(t-t')$$

Solution…

We start with

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{b}(t) = -\left[i\omega_{\text{m}} + \left(\gamma+\gamma_{\text{opt}}\right)\right]\hat{b}(t) + \cdots - \sqrt{2\gamma}\hat{b}_{\text{in}}(t)$$

which can be formally solved to yield

$$\hat{b}(t) = \cdots - \sqrt{2\gamma}\int_{-\infty}^{t} \mathrm{d}\tau\, e^{-[i\omega_{\text{m}}+(\gamma+\gamma_{\text{opt}})](t-\tau)}\hat{b}_{\text{in}}(\tau)$$

But then, the contribution of this term to the occupation number is

$$\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle = 2\gamma\int_{-\infty}^{t}\mathrm{d}\tau\int_{-\infty}^{t}\mathrm{d}\tau'\, e^{-[-i\omega_{\text{m}}+(\gamma+\gamma_{\text{opt}})](t-\tau)}e^{-[i\omega_{\text{m}}+(\gamma+\gamma_{\text{opt}})](t-\tau')}\langle\hat{b}_{\text{in}}^\dagger(\tau)\hat{b}_{\text{in}}(\tau')\rangle$$

Using the given information, we may reduce the above double integral to a single integral:

$$\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle = 2\gamma n\int_{-\infty}^{t}\mathrm{d}\tau\, e^{-[-i\omega_{\text{m}}+(\gamma+\gamma_{\text{opt}})](t-\tau)}e^{-[i\omega_{\text{m}}+(\gamma+\gamma_{\text{opt}})](t-\tau)} = 2\gamma n\int_{-\infty}^{t}\mathrm{d}\tau\, e^{-2(\gamma+\gamma_{\text{opt}})(t-\tau)}$$
$$= 2\gamma n\, e^{-2(\gamma+\gamma_{\text{opt}})t}\int_{-\infty}^{t}\mathrm{d}\tau\, e^{2(\gamma+\gamma_{\text{opt}})\tau} = \frac{\gamma}{\gamma+\gamma_{\text{opt}}}n$$

Task 3

A similar procedure can be performed but taking into account the optical noise terms only. In this case, we approximate the optical field to a zero-temperature white-noise field:

$$\langle\hat{a}_{\text{in}}^\dagger(t)\hat{a}_{\text{in}}(t')\rangle = 0, \qquad \langle\hat{a}_{\text{in}}(t)\hat{a}_{\text{in}}^\dagger(t')\rangle = \delta(t-t')$$

However, because of the form of the optical noise terms, the situation now is a bit more complicated than it was previously. Nevertheless, your task is to obtain the contribution to $\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle$ given by the optical noise terms:

$$\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle = \frac{G^2}{\gamma+\gamma_{\text{opt}}}\frac{\left(\gamma+\gamma_{\text{opt}}\right)+\kappa}{(\omega_{\text{m}}-\Delta)^2 + \left[\left(\gamma+\gamma_{\text{opt}}\right)+\kappa\right]^2}$$

You may also need to use one trick:

$$\int_{-\infty}^{t}\mathrm{d}t'\int_{-\infty}^{t}\mathrm{d}t''\int_{-\infty}^{t'}\mathrm{d}\tau\int_{-\infty}^{t''}\mathrm{d}\tau' f(t',t'',\tau,\tau')\,\delta(\tau-\tau') = \int_{-\infty}^{t}\mathrm{d}t'\int_{-\infty}^{t}\mathrm{d}t''\int_{-\infty}^{\min\{t',t''\}}\mathrm{d}\tau\, f(t',t'',\tau,\tau)$$
$$= \int_{-\infty}^{t}\mathrm{d}t'\int_{-\infty}^{t'}\mathrm{d}t''\int_{-\infty}^{t''}\mathrm{d}\tau\, f(t',t'',\tau,\tau) + \int_{-\infty}^{t}\mathrm{d}t''\int_{-\infty}^{t''}\mathrm{d}t'\int_{-\infty}^{t'}\mathrm{d}\tau\, f(t',t'',\tau,\tau)$$

Solution…

Once again, we may write

$$\hat{b}(t) = -iG\sqrt{2\kappa} \int_{-\infty}^{t} \mathrm{d}t'\, e^{-[i\omega_\mathrm{m}+(\gamma+\gamma_\mathrm{opt})](t-t')} \left[\int_{-\infty}^{t'} \mathrm{d}\tau\, e^{(i\Delta-\kappa)(t'-\tau)} \hat{a}_\mathrm{in}(\tau) + \int_{-\infty}^{t'} \mathrm{d}\tau\, e^{(-i\Delta-\kappa)(t'-\tau)} \hat{a}_\mathrm{in}^{\dagger}(\tau) \right] + \cdots$$

Because of the above relations, the only term that contribute to the occupation number is

$$\langle \hat{b}^{\dagger}(t)\hat{b}(t)\rangle = 2G\kappa \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t} \mathrm{d}t'' \int_{-\infty}^{t'} \mathrm{d}\tau \int_{-\infty}^{t''} \mathrm{d}\tau'\, e^{-[-i\omega_\mathrm{m}+(\gamma+\gamma_\mathrm{opt})](t-t')}\, e^{-[i\omega_\mathrm{m}+(\gamma+\gamma_\mathrm{opt})](t-t'')} \times e^{(i\Delta-\kappa)(t'-\tau)}\, e^{(-i\Delta-\kappa)(t''-\tau')} \langle \hat{a}_\mathrm{in}(\tau)\hat{a}_\mathrm{in}^{\dagger}(\tau')\rangle + \cdots$$

To work this out, it is best to group the exponentials by variable:

- $e^{-2(\gamma+\gamma_\mathrm{opt})t}$
- $e^{-[i\omega_\mathrm{m}-i\Delta-(\gamma+\gamma_\mathrm{opt})-\kappa]t'}$
- $e^{[i\omega_\mathrm{m}-i\Delta+(\gamma+\gamma_\mathrm{opt})-\kappa]t''}$
- $e^{i\Delta(\tau-\tau')}e^{\kappa(\tau+\tau')}$

The first thing to do is get rid of τ' by substituting in the delta function and performing the integral. We can then use the trick outlined above:

$$\begin{aligned}
&\langle \hat{b}^{\dagger}(t)\hat{b}(t)\rangle \\
&= 2G^2\kappa\, e^{-2(\gamma+\gamma_\mathrm{opt})t} \Bigg\{ \int_{-\infty}^{t} \mathrm{d}t'\, e^{-[i\omega_\mathrm{m}-i\Delta-(\gamma+\gamma_\mathrm{opt})-\kappa]t'} \int_{-\infty}^{t'} \mathrm{d}t''\, e^{[i\omega_\mathrm{m}-i\Delta+(\gamma+\gamma_\mathrm{opt})-\kappa]t''} \int_{-\infty}^{t''} \mathrm{d}\tau\, e^{2\kappa\tau} \\
&+ \int_{-\infty}^{t} \mathrm{d}t''\, e^{[i\omega_\mathrm{m}-i\Delta+(\gamma+\gamma_\mathrm{opt})-\kappa]t''} \int_{-\infty}^{t''} \mathrm{d}t'\, e^{-[i\omega_\mathrm{m}-i\Delta-(\gamma+\gamma_\mathrm{opt})-\kappa]t'} \int_{-\infty}^{t'} \mathrm{d}\tau\, e^{2\kappa\tau} \Bigg\}
\end{aligned}$$

Notice how the second term in the braces is the complex conjugate of the first. Thus,

$$\begin{aligned}
\langle \hat{b}^{\dagger}(t)\hat{b}(t)\rangle &= G^2\, e^{-2(\gamma+\gamma_\mathrm{opt})t} \Bigg\{ \int_{-\infty}^{t} \mathrm{d}t'\, e^{-[i\omega_\mathrm{m}-i\Delta-(\gamma+\gamma_\mathrm{opt})-\kappa]t'} \int_{-\infty}^{t'} \mathrm{d}t''\, e^{[i\omega_\mathrm{m}-i\Delta+(\gamma+\gamma_\mathrm{opt})+\kappa]t''} \\
&\qquad + \mathrm{c.c.} \Bigg\} \\
&= G^2\, e^{-2(\gamma+\gamma_\mathrm{opt})t} \left[\frac{1}{i\omega_\mathrm{m}-i\Delta+(\gamma+\gamma_\mathrm{opt})+\kappa} \int_{-\infty}^{t} \mathrm{d}t'\, e^{2(\gamma+\gamma_\mathrm{opt})t'} + \mathrm{c.c.} \right] \\
&= \frac{G^2}{2(\gamma+\gamma_\mathrm{opt})} \left[\frac{1}{i\omega_\mathrm{m}-i\Delta+(\gamma+\gamma_\mathrm{opt})+\kappa} + \mathrm{c.c.} \right] \\
&= \frac{G^2}{\gamma+\gamma_\mathrm{opt}} \frac{(\gamma+\gamma_\mathrm{opt})+\kappa}{(\omega_\mathrm{m}-\Delta)^2+\left[(\gamma+\gamma_\mathrm{opt})+\kappa\right]^2}
\end{aligned}$$

Task 4

Now it's time for something easy. Making use of the following facts:

- There is no cross-correlation between $\hat{a}_{\mathrm{in}}$ and $\hat{b}_{\mathrm{in}}$, so the above two terms simply add
- Work on the red sideband, i.e., $\Delta = -\omega_{\mathrm{m}}$
- Assume the resolved sideband regime, i.e., $\omega_{\mathrm{m}} \gg \kappa \gg \gamma, \gamma_{\mathrm{opt}}$

By defining an effective optically-induced phonon number, $n_{\mathrm{opt}} = \left(\frac{\kappa}{2\omega_{\mathrm{m}}}\right)^2$, show that you obtain the simple expression:

$$\langle \hat{b}^\dagger(t)\hat{b}(t)\rangle = \frac{\gamma\, n + \gamma_{\mathrm{opt}}\, n_{\mathrm{opt}}}{\gamma + \gamma_{\mathrm{opt}}}$$

Solution…

$$\langle \hat{b}^\dagger(t)\hat{b}(t)\rangle = \frac{\gamma}{\gamma + \gamma_{\mathrm{opt}}} n + \frac{G^2}{\gamma + \gamma_{\mathrm{opt}}} \frac{\kappa}{(2\omega_{\mathrm{m}})^2} = \frac{\gamma}{\gamma + \gamma_{\mathrm{opt}}} n + \frac{\gamma_{\mathrm{opt}}}{\gamma + \gamma_{\mathrm{opt}}}\left(\frac{\kappa}{2\omega_{\mathrm{m}}}\right)^2 = \frac{\gamma\, n + \gamma_{\mathrm{opt}}\, n_{\mathrm{opt}}}{\gamma + \gamma_{\mathrm{opt}}}$$

Notes

In the limit where our coupling constant G is very large, we have $\gamma_{\mathrm{opt}} \gg \gamma\, n \gg \gamma$, such that $\langle \hat{b}^\dagger(t)\hat{b}(t)\rangle \approx \left(\frac{\kappa}{2\omega_{\mathrm{m}}}\right)^2$. Under resolved-sideband conditions, this is the minimum possible occupation number of the mechanical oscillator. Note that this contribution to $\langle \hat{b}^\dagger(t)\hat{b}(t)\rangle$ arises entirely from the term in $\frac{\mathrm{d}}{\mathrm{d}t}\hat{b}$ proportional to $\hat{a}^\dagger$, and can therefore be traced to the part of the interaction Hamiltonian proportional to $\left(\hat{a}\hat{b} + \hat{a}^\dagger\hat{b}^\dagger\right)$. In other words, we can state that *the optical heating is due to amplified vacuum fluctuations*.

How else can we interpret this optically-induced photon number? If we delve deeper into the theory, it turns out that this phonon number depends on the density of states of the optical field at a frequency ω_{m} lower than that of the driving field. This density of states tells me how likely I am to scatter a driving photon into a *lower frequency* state in the cavity field, thereby heating up the oscillator by conservation of energy. Cavity spectra are Lorentzian, so this probability is never zero. However, by making $\omega_{\mathrm{m}} \gg \kappa$, we may make this probability (and, consequently, n_{opt}) very small. One way around this issue is the dissipative coupling mechanism that will be mentioned in Lecture IV. For more details, see [F. Marquardt, J. P. Chen, A. A. Clerk, and S. M. Girvin, Phys. Rev. Lett. **99**, 093902 (2007)] and [A. Xuereb, R. Schnabel, and K. Hammerer, Phys. Rev. Lett. **107**, 213604 (2011)].