Optomechanics I

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Credits

This series of lectures draws heavily from a lecture by Klemens Hammerer, called “Quantum Optomechanics” and delivered at the QLNO Summer School in August 2010.
Overview of this lecture

- Basic ideas: Photon momentum and radiation pressure
- Links with cold atoms
- Our workhorse: Cavity optomechanics
  - Unitary dynamics: The system Hamiltonian
  - Opening the system: Cavity losses and mechanical damping
Radiation pressure

- Before we can talk about optomechanics: Radiation pressure
- Radiation pressure is simply the capability of electromagnetic radiation to push things around
- On an intuitive level, one can understand radiation pressure as arising from the conservation of momentum:

\[
p_{\text{rad}} = \hbar k
\]

\[
p_{\text{rad}} = -\hbar k
\]

\[
p_{\text{mir}} = 2\hbar k
\]
Radiation pressure

- The use of radiation pressure *in a controlled setting* is a late 20\textsuperscript{th} century phenomenon
- However, astronomers have been observing it for hundreds of years
- Indeed, on galactic scales it can be quite influential
- Closer to home, comet tails also exhibit its effects
Radiation pressure
Radiation pressure
Let’s start from the beginning...

- Probably one of the first mentions of the mechanical effects of light on massive objects was by Einstein at the 81st Meeting of the Society of Natural Scientists and Medics (Salzburg, 1909):

  “On the development of our views concerning the nature and constitution of radiation”

- Einstein goes on to derive a force of friction acting on a mirror that is moving inside a blackbody radiation field
Einstein’s thought experiment

- Consider a chamber containing an ideal gas at temperature $T$
- Inside the chamber we place a harmonically bound, perfectly reflecting mirror
- Blackbody radiation is also present, of course
Einstein’s thought experiment

- Each reflected wave has a Doppler shift:
  \[ k' = -k \left(1 - \frac{2v}{c}\right) \]

- This yields a radiation pressure force
  \[ F_{rp} = \frac{2P}{c} \left(1 - \frac{v}{c}\right) \]

- Therefore, radiation provides friction:
  \[ \dot{p} = -\gamma p, \quad \gamma = \frac{2P}{mc^2} \]
Let’s start from the beginning...

- So the gas molecules transfer their energy to the mechanical plate by hitting it...
- ... and the motion is then damped by the radiation
- This results in a continuous transfer of energy from the gas to the radiation, and no equilibrium is possible
- Einstein understood that the way out was to consider that light, like molecules, comes in discrete packets
- This forces the process to stop extracting energy from the mirror beyond a certain point
Let’s start from the beginning...

- The result of this consideration is a beautiful formula for the average-square momentum transferred due to these “fluctuations”:

\[
\Delta^2 = \frac{1}{c} \left[ h\nu + \frac{c^3}{8\pi} \frac{\varphi^2}{\nu^2} \right] dv df.
\]

- The first term is explained by localised particles moving with an energy \( h\nu \); this term is dominant for low energy density.

- The second term can be explained by wave-like considerations.

- Radiation pressure has been used to discuss the quantum nature both of light (1909) and of massive matter (optomechanics).
Cold atoms

- Between Einstein and optomechanics as I shall discuss it lies one further intermediate step

- Starting in the 1970s, experiments were conducted to trap and cool atoms using the effects of radiation pressure
Optical molasses
The magneto-optical trap
Bose–Einstein Condensation
Optomechanics and cold ions

- We will concern ourselves with how light interacts with massive, vibrating mirrors inside cavities.
- It turns out that this system bears more than a passing resemblance to the use of light to cool trapped ions.
A connection with trapped ions

- Consider a two-level atom in a harmonic trap (frequency $\omega_m$)
- The energy level structure of the entire system is composed of two harmonic-oscillator ladders separated by some energy $\hbar \omega_a$
- Call the ground state levels $|n, g\rangle$ and the excited states $|n, e\rangle$
- We can use a laser to drive the $|n, g\rangle \rightarrow |n - 1, e\rangle$ transition
- Spontaneous emission takes the atom down to $|n - 1, g\rangle$
- Result: Atomic motion is cooled
A connection with trapped ions
Cavity optomechanics

- We introduce our workhorse: The simplest cavity optomechanical system
Cavity optomechanics: Parameters

- $L$: Length of the cavity when the mirror is in its rest position
- $\omega_m$: Oscillation frequency of the mirror
- $\omega_c$: Resonance frequency of the cavity
- $m$: Mass of the mirror
- $x_{zpt} = \sqrt{\frac{\hbar}{m\omega_m}}$: Size of the zero-point wave function of the mirror
Cavity optomechanics: Hamiltonian

- First, consider the uncoupled optical and mechanical systems
- For the optical field, the Hamiltonian reads
  \[ \hat{H}_{\text{opt}} = \hbar \omega_c \hat{a}^\dagger \hat{a} \]
- We denote by \( \hat{a} \) the annihilation operator for the optical field
- We have neglected the \( \frac{1}{2} \hbar \omega_c \), since this will not affect our results
Cavity optomechanics: Hamiltonian

- For the mechanical subsystem,
  \[ \hat{H}_{\text{mech}} = \frac{1}{2} m \omega_m^2 \hat{x}^2 + \frac{\hat{p}^2}{2m} \]

- Define an annihilation operator \( \hat{b} \) such that \( \hat{x} = x_{zpt} (\hat{b} + \hat{b}^\dagger) / \sqrt{2} \) and \( \hat{p} = m \omega_m x_{zpt} (\hat{b} - \hat{b}^\dagger) / (i\sqrt{2}) \)

- Then, \( \hat{H}_{\text{mech}} = \hbar \omega_m \hat{b}^\dagger \hat{b} \)

- Again, we have neglected the \( \frac{1}{2} \hbar \omega_m \)
Cavity optomechanics: Hamiltonian

- Next, we need to see how the two systems talk to one another
- We note that $\omega_c = \frac{2\pi nc}{L}$, where $n = 1, 2, 3, ...$
- What happens when the mirror moves by a distance $x$? $L \to L + x!$
Cavity optomechanics: Hamiltonian

- Suppose, now that $|x| \ll L$, and expand to first order:
  \[ \omega_c \rightarrow \omega_c - \frac{\omega_c}{L} x \]
- Replace $x$ by $\hat{x}$:
  \[ \omega_c \rightarrow \omega_c - \frac{\omega_c}{L} \hat{x} = \omega_c - \frac{\omega_c}{\sqrt{2L}} x_{zpt}(\hat{b} + \hat{b}^\dagger) \]
Cavity optomechanics: Hamiltonian

- Let us now define the optomechanical coupling strength:

\[ g := \frac{\omega_c}{\sqrt{2L}} x_{\text{zpt}} = \sqrt{\frac{\hbar}{2m\omega_m}} \frac{\omega_c}{L} \]

- Note that \( g \) is dimensionally a frequency, and can be compared to \( \omega_m \), etc., as we shall see later.
Cavity optomechanics: Hamiltonian

- Let us get back to our optical Hamiltonian
  \[ \hat{H}_{\text{opt}} \rightarrow \hbar [\omega_c - g(\hat{b} + \hat{b}^\dagger)]\hat{a}^{\dagger}\hat{a} = \hat{H}_{\text{opt}} + \hat{H}_{\text{int}} \]

- The optomechanical interaction Hamiltonian is simply given by
  \[ \hat{H}_{\text{int}} = -\hbar g(\hat{b} + \hat{b}^\dagger)\hat{a}^{\dagger}\hat{a} \]

- Finally, then, the full optomechanical Hamiltonian is
  \[ \hat{H}_{\text{OM}} = \frac{\hbar \omega_c}{\hat{H}_{\text{opt}}} \hat{a}^{\dagger}\hat{a} + \frac{\hbar \omega_m}{\hat{H}_{\text{mech}}} \hat{b}^{\dagger}\hat{b} - \frac{\hbar g}{\hat{H}_{\text{int}}} (\hat{b} + \hat{b}^\dagger)\hat{a}^{\dagger}\hat{a} \]
Cavity optomechanics: Hamiltonian

- This derivation is not entirely rigorous
- More rigorous derivations are available in the literature, but they go well beyond the scope of this course
- For further detail, consult:
  - ... and several other works
- The conclusions, however, are unchanged for most situations
Cavity optomechanics: Hamiltonian

- How does one measure $g$ in the laboratory?
- It is easy to see that
  \[
g = -\frac{x_{zpt} \, d\omega_c}{\sqrt{2}} \frac{d\omega_c}{dx}
\]
- Experiments always have control over the position $x$ to a very good degree
- A measurement of $g$ is therefore possible by looking at how the frequency of the cavity mode changes as a function of $x$
Cavity optomechanics: Hamiltonian

[J. D. Thompson, et al., Nature 452, 72 (2008)]
Cavity optomechanics: Hamiltonian

- These measurements are for a mirror *inside* the cavity.
- A calculation for $g$ in this case proceeds along different lines but the result is still quite similar.
- Notice how there are places where $g$ is maximal and others where $g = 0$.
- At the latter points, the lowest-order optomechanical coupling is of the form $\hat{a}^\dagger \hat{a} \hat{x}^2$ ("quadratic") rather than $\hat{a}^\dagger \hat{a} \hat{x}$ ("linear").
- We should have time to look briefly into this later, but for now we shall focus on the more common linear case.
Cavity optomechanics: Hamiltonian

- Summary so far:
  - A simple optomechanical system is described by the Hamiltonian
    \[ \hat{H}_{\text{OM}} = \hbar \omega_c \hat{a} \hat{a}^\dagger + \hbar \omega_m \hat{b} \hat{b}^\dagger - \hbar g (\hat{b}^\dagger \hat{b}) \hat{a}^\dagger \hat{a} \]
  - The optomechanical coupling strength is
    \[ g = \sqrt{\frac{\hbar \omega_c}{2m\omega_m L}} \]
  - This Hamiltonian can be used to generate the unitary dynamics of the system:
    \[ \dot{\rho} = \frac{i}{\hbar} [\hat{H}_{\text{OM}}, \rho] = \frac{i}{\hbar} (\hat{H}_{\text{OM}} \rho - \rho \hat{H}_{\text{OM}}) \]
- It is now time to make the system talk to the outside world
Cavity optomechanics: Open systems

- There are three ways in which our system interacts with the universe
  - First, we *drive* the system
  - Second, the cavity field *leaks* through the mirror
  - Third, the mechanical vibrations *leak* to the mirror’s support

- We shall now look at each of these in turn
Cavity optomechanics: Driving

- Driving the cavity is very simple
- We must add a forcing term to our Hamiltonian
  \[ \hat{H}_d = -i\hbar(\varepsilon^* e^{i\omega_d t} \hat{a} - \varepsilon e^{-i\omega_d t} \hat{a}^\dagger) \]
- \(\omega_d\) is the frequency of the driving field (assumed monochromatic)
- What is the effect of this term?
- Let us use \(\langle \hat{a} \rangle = \frac{d}{dt} \langle \hat{a} \rangle = \text{Tr}\{\hat{a} \rho\} = -\frac{i}{\hbar} \text{Tr}\{\hat{a} [\hat{H}_d, \rho]\} = \frac{i}{\hbar} \langle [\hat{H}_d, \hat{a}] \rangle\)
- Therefore, \(\langle \hat{a} \rangle = \varepsilon e^{-i\omega_d t}\)
Cavity optomechanics: Driving

- We often make a very useful transformation
- Basically, we want to get rid of that annoying $e^{-i\omega_d t}$!
- This is done by using the unitary transformation
  $$U = \exp(i\omega_d \hat{a}^\dagger \hat{a} t)$$
- The transformed system obeys the Hamiltonian
  $$\hat{H} = U (\hat{H}_{OM} + \hat{H}_d) U^\dagger - \hbar \omega_d \hat{a}^\dagger \hat{a}$$
Cavity optomechanics: Driving

- This particular transformation is effected in two steps:
  - Replace: $\hat{a} \rightarrow \hat{a} e^{-i\omega_d t}$
  - Subtract $\hbar \omega_d \hat{a}^\dagger \hat{a}$ from the total Hamiltonian

- We then obtain:
  \[
  \hat{H} = -\hbar \Delta \hat{a}^\dagger \hat{a} + \hbar \omega_m \hat{b}^\dagger \hat{b} - \hbar g (\hat{b} + \hat{b}^\dagger) \hat{a}^\dagger \hat{a} - i\hbar (\varepsilon^* \hat{a} - \varepsilon \hat{a}^\dagger)
  \]

- The *detuning* is defined as the difference between the cavity frequency and the driving frequency
  \[
  \Delta := \omega_d - \omega_c
  \]
Cavity optomechanics: Driving

- The detuning is a very important parameter in optomechanics
- We will need to refer to it, so let us establish some nomenclature
  - **Red detuning** refers to the situation when $\Delta < 0$; in this situation the driving laser has a smaller frequency than the cavity resonance (i.e., it is closer to the red part of the spectrum)
  - **Blue detuning** refers to the situation when $\Delta > 0$; in this situation the driving laser has a larger frequency than the cavity resonance (i.e., it is closer to the blue part of the spectrum)
- This convention is not universal!
Cavity optomechanics: Damping

- The second component of our open system is the damping (or leaking) of excitations to the environment.
- For the optical system, this happens through the leakage of photons through the mirrors (which are never perfectly reflective).
- For the mechanical system, phonons travel through the support to or from the rest of the laboratory.
- We shall handle these two in exactly the same manner.
Cavity optomechanics: Damping

- We don’t have any space to go into the hows and whys of things here.
- I must quote some results which I do not have the time to prove.
- In 1968, Göran Lindblad showed that the most general non-unitary part of the master equation must have the form:

\[
\Gamma \mathcal{D}_\delta[\rho] := \Gamma [2\delta \rho \delta^\dagger - (\rho \delta^\dagger \delta + \delta^\dagger \delta \rho)]
\]

- Here, \( \Gamma > 0 \) is some rate and \( \delta \) is an operator of the system.
Cavity optomechanics: Damping

- In the case of the optical field, we set:
  - $\Gamma \rightarrow \kappa$, which is the rate at which the complex amplitude of the cavity field is damped
  - $\hat{\sigma} \rightarrow \hat{a}$, which means that the cavity field is damped through its annihilation operators (i.e., the environment acts to annihilate photons in the cavity)
- The above assumption corresponds to saying that the photon field outside the cavity is at 0 K, which is a good approximation for optical fields
- In the case of microwaves, we have additional terms similar to the mechanical system
Cavity optomechanics: Damping

- We therefore have the first Liouvillian of our system:
  \[ \mathcal{L}_{\text{opt}}[\rho] = \kappa [2\hat{a}\rho\hat{a}^\dagger - (\rho\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}\rho)] \]

- This term, and the ones we will discuss soon, are added to the master equation:
  \[ \dot{\rho} = -\frac{i}{\hbar} [\hat{H}, \rho] + \mathcal{L}_{\text{opt}}[\rho] + \ldots \]
Cavity optomechanics: Damping

- Let us see what effect this term has on the field expectation values
- Consider:
  \[ \text{Tr}\left\{ \hat{a}[2\hat{a}\rho\hat{a}^\dagger - (\rho\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}\rho)] \right\} = \text{Tr}\left\{ (\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger)\rho \right\} = -\langle \hat{a} \rangle \]
- Combining both Hamiltonian and Liouvillian parts \((g = 0)\), we get
  \[ \langle \hat{a} \rangle = (i\Delta - \kappa)\langle \hat{a} \rangle + \varepsilon \]
- At steady state, \(\langle \hat{a} \rangle = 0\) and
  \[ \langle \hat{a} \rangle = -\frac{\varepsilon}{i\Delta - \kappa} \]
- The situation is, of course, more complicated when \(g \neq 0\)
Cavity optomechanics: Damping

- For the mechanical field, we have two different Liouvillians

- One, like $\mathcal{L}_{\text{opt}}[\rho]$, describes the leaking of phonons from the system to the outside world

- For this term we set:
  - $\Gamma \rightarrow \gamma(n + 1)$, which is the amplitude decay rate of the mechanical oscillations multiplied by the mean number of excitations in the support plus 1
  - $\hat{o} \rightarrow \hat{b}$, similarly to the optical case
Cavity optomechanics: Damping

- The mean number of excitations in the support is given by the factor
  \[ n = \frac{1}{\exp \left( \frac{\hbar \omega_m}{k_B T} \right) - 1} \]

- \( k_B \) is the Boltzmann constant and \( T \) is the environment temperature

- For large temperatures (\( T \gg \frac{\hbar \omega_m}{k_B} \)), we may estimate
  \[ n \approx \frac{k_B T}{\hbar \omega_m} \]

- When \( T \ll \frac{\hbar \omega_m}{k_B} \), \( n \) is exponentially suppressed (\( n \to 0 \) for the optical field)
Cavity optomechanics: Damping

- The second term describes the leaking of phonons from the outside world into the system.
- For this term we set:
  - $\Gamma \rightarrow \gamma n$, which is the amplitude decay rate of the mechanical oscillations multiplied by the mean number of excitations in the support.
  - $\delta \rightarrow \hat{b}^\dagger$, which means that this Liouvillian creates excitations.
Cavity optomechanics: Damping

- Putting these two terms together, we find
  \[ \mathcal{L}_{\text{mech}}[\rho] = \gamma (n + 1) [2 \hat{b} \rho \hat{b}^\dagger - (\rho \hat{b}^\dagger \hat{b} + \hat{b}^\dagger \hat{b} \rho)] + \gamma n [2 \hat{b}^\dagger \rho \hat{b} - (\rho \hat{b} \hat{b}^\dagger + \hat{b} \hat{b}^\dagger \rho)] \]

- Finally, the Liouvillian of the entire system is, quite simply,
  \[ \mathcal{L}[\rho] = \mathcal{L}_{\text{opt}}[\rho] + \mathcal{L}_{\text{mech}}[\rho] \]
Cavity optomechanics: All together!

- To sum up, we have
  \[ \dot{\rho} = -\frac{i}{\hbar} [\hat{H}, \rho] + \mathcal{L}[\rho] \]
- ... where the Hamiltonian is given by \( (\Delta = \omega_d - \omega_c) \)
  \[ \hat{H} = -\hbar \Delta \hat{a}^\dagger \hat{a} + \hbar \omega_m \hat{b}^\dagger \hat{b} - \hbar g (\hat{b} + \hat{b}^\dagger) \hat{a}^\dagger \hat{a} - i\hbar (\mathcal{E}^* \hat{a} - \mathcal{E} \hat{a}^\dagger) \]
- ... and the non-unitary terms
  \[ \mathcal{L}[\rho] = \kappa \mathcal{D}_\hat{a}[\rho] + \gamma (n + 1) \mathcal{D}_\hat{b}[\rho] + \gamma n \mathcal{D}_{\hat{b}^\dagger}[\rho] \]
- ... with
  \[ \mathcal{D}_\hat{\phi}[\rho] = 2\hat{\phi}\rho\hat{\phi}^\dagger - (\rho\hat{\phi}^\dagger\hat{\phi} + \hat{\phi}^\dagger\hat{\phi}\rho) \]
End of Lecture I

Any questions?